Deformations of monic polynomial matrices.
Analysis of perturbation of eigenvalues

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Abstract—Let \( P(\lambda) = \sum_{i=0}^{k} \lambda^{i} A_{i}(p) \) be a family of monic polynomial matrices smoothly dependent on a vector of real parameters \( p = (p_{1}, \ldots, p_{n}) \). In this work we study behavior of a multiple eigenvalue of the monic polynomial family \( P(\lambda) \) as well as we study behavior of a simple eigenvalue of a family of 1-degree singular polynomial matrices representing families of singular linear systems.

Keywords—Polynomial matrix, Eigenvalues, Perturbation, Polynomial matrix, Eigenvalues, Perturbation.

I. INTRODUCTION

Given a polynomial matrix \( P(\lambda) = \sum_{i=0}^{k} \lambda^{i} A_{i} \), where \( A_{i} \) are square matrices over real or complex field, it is an important question from both the theoretical and the practical points of view to know how the eigenvalues and eigenvectors change when the elements of \( P(\lambda) \) are subjected to small perturbations.

Eigenvalue problem for polynomial matrices \( P(\lambda) v = 0 \), appears (among many other applications) modeling physical and engineering problems by means systems of \( k \)-order linear ordinary differential equations. The values of eigenvalues can correspond among others, to frequencies of vibration, critical values of stability parameters, or energy levels of atoms.

The eigenvalues of some matrices are sensitive to perturbations, it is well known that the eigenvalues of monic polynomial matrices are continuous functions of the entries of the matrix coefficients of the polynomial, but Small changes in the matrix elements can lead to large changes in the multiplicity of eigenvalues. For example a little perturbation of the matrix \( (\frac{1}{2} \ 1) \) as \( (\frac{1}{2} \ 1+\epsilon) \) the double eigenvalue \( \lambda = 0 \) is perturbed to two different eigenvalues \( \lambda = \pm \sqrt{\epsilon} \) changing completely the structure of the polynomial matrix. Obviously if we consider the perturbation \( (\frac{1}{0} \ \frac{1}{\lambda}) \) there are not changes in the structure.

Given a square complex matrix \( A \), it is an important question from both the theoretical and the practical points of view to know how the eigenvalues and eigenvectors change when the elements of \( A \) are subjected to small perturbations. The usual formulation of the problem introduces a perturbation parameter \( \epsilon \) belonging to some neighborhood of zero, and writes the perturbed matrix as \( A + \epsilon B \) for an arbitrary matrix \( B \). In this situation, it is well known [15] section II.1.2, that each eigenvalue or eigenvector of \( A + \epsilon B \) admits an expansion in fractional powers of \( \epsilon \), whose zero-th order term is an eigenvalue or eigenvector of the unperturbed matrix \( A \).

In this paper, in section I we present an overview over polynomial matrices \( P(\lambda) \) and the analysis of perturbation of simple eigenvalue \( \lambda_{0} \) of \( P(\lambda) \) such that 0 is a simple eigenvalue of the linear map \( P(\lambda_{0}) \). In section 3, inspired by the work of Arnold [2] on versal deformations of square matrices, we derive versal deformations providing a decomposition of arbitrary perturbation into tangential orthogonal spaces of the set of equivalent polynomial matrices.

In section 4, we study the perturbation of a multiple eigenvalue with a simple eigenvector of a monic polynomial matrix smoothly depending on parameters. Finally, in sections 5 and 6 we generalize the results to analysis of perturbation of simple eigenvalues of standard systems and singular systems.

The study of behavior of simple and multiple eigenvalues of a matrix depending smoothly of parameters has a great interest for its many applications. Perturbation theory for eigenvalues and eigenvectors of regular pencils is well established see [1],[17] for example and for vibrational systems in [16]. In this paper we extend some of these results to polynomial matrices.

II. PRELIMINARIES

A square polynomial matrix of size \( n \) and degree \( k \) is a polynomial of the form

\[
P(\lambda) = \sum_{i=0}^{k} \lambda^{i} A_{i}, \quad A_{0}, \ldots, A_{k} \in M_{n}(\mathbb{F}),
\]

where \( \mathbb{F} \) is the field of real or complex numbers. Our focus is on monic polynomial matrices. A square polynomial matrix \( P(\lambda) \) is said to be monic if \( A_{k} = I_{n} \) is identically. The polynomial matrix (1) naturally arises associated with linear systems of differential equations

\[
A_{k}x^{(k)}(t) + A_{k-1}x^{(k-1)}(t) + \ldots + A_{1}x^{1}(t) + A_{0}x(t) = f(t)
\]

where \( x(t) \) is a vector-valued function (unknown) with \( n \) coordinates, \( x^{(j)}(t) \) denotes the \( j \)-th derivative of \( x(t) \) and \( f(t) \) is another vector-valued function with \( n \) coordinates. Of particular relevance is the case of linear systems of second order, appearing in many engineering applications.

The eigenvalues of a polynomial matrix \( P(\lambda) \) are the zeros of the \( nk \)-degree scalar polynomial \( \det P(\lambda) \).

Let \( \lambda_{0} \) be an eigenvalue of polynomial matrix \( P(\lambda) \), then there exists a vector \( v_{0} \neq 0 \) such that \( P(\lambda_{0})v_{0} = 0 \), this vector is called an eigenvector.

We will call a Jordan chain of length \( k+1 \) for \( P(\lambda) \) corresponding to complex number \( \lambda_{0} \) to the sequence of \( n \)-dimensional vectors \( v_{0}, \ldots, v_{k} \) such that

\[
\sum_{i=0}^{k} \frac{1}{i!} P^{(i)}(\lambda_{0})v_{i-\ell} = 0, \quad i = 0, \ldots, k
\]

where \( P^{(i)}(\lambda) \) denotes the \( \ell \)-derivative of \( P(\lambda) \) with respect the variable \( \lambda \). If \( \lambda_{0} \) is an eigenvalue there exists a Jordan chain of length at least 1 formed by the eigenvector.

Let \( \lambda_{0} \) be an eigenvalue of \( P(\lambda) \), then \( \det P(\lambda_{0}) = \det P(\lambda) = 0 \), so \( \lambda_{0} \) is an eigenvalue of \( P(\lambda) \). For this eigenvalue there exists an eigenvector \( u_{0} \), that is \( P(\lambda_{0})u_{0} = 0 \), equivalently \( u_{0}^{T}P(\lambda_{0}) = 0 \). The vector \( u_{0} \) is called left eigenvector corresponding to the eigenvalue \( \lambda_{0} \) of \( P(\lambda) \).

For more information see [5], or [14] for example.

Let \( P(\lambda) = \sum_{i=0}^{k} \lambda^{i} A_{i} \) be now, a polynomial matrix and we assume that the matrices \( A_{i} \) smoothly depend on the vector of real parameters \( p = (p_{1}, \ldots, p_{r}) \). The function \( P(\lambda; p) = \sum_{i=0}^{k} \lambda^{i} A_{i}(p) \) is called a multi-parameter family of polynomial matrices. Eigenvalues of the polynomial matrix function are continuous functions of the vector of parameters. We are going to review the behavior of a simple eigenvalue of the family of polynomial matrices \( P(\lambda; p) \).
Let $\lambda(p)$ be a simple eigenvalue of the polynomial matrix $P(\lambda; p)$. Since $\lambda(p)$ is a simple root of the scalar polynomial $\det P(\lambda)$, we have
\[
\frac{\partial}{\partial \lambda} \det P(\lambda; p) \neq 0.
\]

The expression (4) permit us to make use the implicit function theorem to the equation $\det P(\lambda; p) = 0$, and we observe that the eigenvalue $\lambda(p)$ of the family of polynomial matrices smoothly depends on the vector of parameters, and its derivatives with respect to parameters are
\[
\frac{\partial \lambda(p)}{\partial p_i} = -\frac{\frac{\partial}{\partial p_i} \det P(\lambda; p)}{\frac{\partial}{\partial \lambda} \det P(\lambda; p)}, \quad i = 1, \ldots, r.
\]

Taking into account that $\lambda(p)$ is a simple eigenvalue and that the sum of the lengths of Jordan chains in a canonical set is the multiplicity of the eigenvalue as zero of $\det P(\lambda; p)$, we have that the Jordan chains consist only of the eigenvectors.

The eigenvector $v_0(p)$ corresponding to the simple eigenvalue $\lambda(p)$ is determined up to a nonzero scaling factor $\alpha$. This eigenvector determines a one-dimensional null-subspace of the matrix operator $P(\lambda(p); p)$ smoothly dependent on $p$. Hence, the eigenvector $v_0(p)$ can be chosen as a smooth function of the parameters.

An approximation of the eigenvalues as well of the corresponding eigenvectors by means their derivatives is given by the following result.

**Theorem 1:**
\[
\frac{\partial \lambda}{\partial p_i} = \frac{u_0^T \frac{\partial P(\lambda; p)}{\partial p_i} v_0(p_0)}{u_0^T \frac{\partial P(\lambda; p)}{\partial \lambda} v_0(p_0)},
\]

and
\[
\frac{\partial v_0(p)}{\partial p_i} = -T_0^{-1} \left( \frac{\frac{\partial}{\partial p_i} P'(\lambda; p)}{\frac{\partial}{\partial \lambda} P' \lambda; p} \right) v_0(p_0),
\]

where $T_0 = P(\lambda_0; p_0) + u_0 u_0^T P'(\lambda_0; p_0)$, and
\[
\frac{\partial^2 \lambda}{\partial p_i \partial p_j} = -\frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} \frac{\partial \lambda(p)}{\partial p_i} = -\frac{\partial^2}{\partial p_i \partial p_j} \lambda(p).
\]

Taking $z(\xi) = \alpha(g(\xi), \varphi(\phi(\xi)))$ of (7) of (0) and that $g_0(\xi) = \xi$ and $\varphi_0(\xi) = \xi$, we have that
\[
\frac{\partial z}{\partial p_i} \left( \frac{\partial}{\partial p_i} z(\xi) \right) = a
\]
with
\[
a = \left( u_0^T \frac{\partial}{\partial p_i} P'(\lambda; p) \right) v_0(p_0),
\]

and
\[
b = u_0^T P'(\lambda_0; p_0) v_0(p_0).
\]

Thus, we obtain that $\varphi(\phi(\xi))$ of $x_0$ with the parameter vector $\xi \in \mathbb{R}^d$. Applying the equivalence transformation $g(\xi)$, where $g: \mathbb{U}_0 \to \mathbb{G}$ is a smooth mapping such that $g(0) = e$ is the unit element of $\mathbb{G}$, we get the deformation
\[
z(\xi) = g(\xi), \varphi(\phi(\xi))
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**III. VERSAL DEFORMATIONS**

The Arnold technique of constructing a local canonical form, called versal deformation, of a differentiable family of square matrices under conjugation [2] provide a special parametrization of matrix spaces, which can be effectively applied to perturbation analysis and investigation of complicated objects like singularities and bifurcations in multi-parameter dynamical systems (see [2], [6] among others).

The general notion of versality is the following. Let $\mathbb{M}$ be a differential manifold with the equivalence relation defined by the action $\alpha(g, x)$ (where $x, g \circ x \in \mathbb{M}$ and $g \in G$) of a Lie group $G$. Let us consider a smooth mapping $x: \mathbb{U}_0 \to \mathbb{M}$, where $\mathbb{U}_0$ is a neighborhood of the origin of the space $\mathbb{E}$. The mapping $\varphi(\xi)$ is called deformation of $x_0 = x(0)$ with the parameter vector $\xi \in \mathbb{E}$. Introducing a change of parameters $\phi: \mathbb{U}_0 \to \mathbb{U}_0$, where $\mathbb{U}_0$ is a neighborhood of the origin in $\mathbb{E}$, such that $\phi(0) = 0$, we obtain the deformation $\varphi(\phi(\xi))$ of $x_0$ with the parameter vector $\xi \in \mathbb{U}_0 \subset \mathbb{E}$. Applying the equivalence transformation $g(\xi)$, where $g: \mathbb{U}_0 \to \mathbb{G}$ is a smooth mapping such that $g(0) = e$ is the unit element of $\mathbb{G}$, we get the deformation
\[
z(\xi) = g(\xi), \varphi(\phi(\xi))
\]
Proposition 2: Let $\alpha_{\epsilon}(x) : T_{x_0} \rightarrow \mathcal{P}_k(\lambda)$ be the differential of the polynomial matrix $A_k(\lambda)$ at the unit element $I$. Then

\[
d\alpha_{\epsilon}(S) = (A_{k-1}S - SA_{k-1}, \ldots, A_0S - SA_0) \in \mathcal{P}_k(\lambda), S \in T_{x_0} G.
\]

Remark 1: It is well-known that the map $d\alpha_{\epsilon}(x)$ provides a simple description of the tangent space $T_{x_0} \mathcal{O}(x_0)$.

Proposition 3:

\[
T_{x_0} \mathcal{O}(x_0) = \text{Im} \ d\alpha_{\epsilon}(x) \in \mathcal{P}_k(\lambda).
\]

Hermitian $\mathcal{P}_k(\lambda)$ and $T_{x_0} \mathcal{G}$ we will deal with in this paper are the following ones:

\[
(x_1, x_2) = \text{tr}(A_{k-1}A_{k-1}^* + \ldots \text{tr}(A_0, A_{k-2}),
\]

where $A^*$ denotes the conjugate transpose of the matrix $A$.

Using the hermitian property (11) is easily to deduce a description of $T_{x_0} \mathcal{O}(x_0)$ for $x_0 \in \mathcal{P}_k(\lambda)$.

Proposition 4: Let $x_0 = (A_{k-1}, \ldots, A_0) \in \mathcal{P}_k(\lambda)$. Then $(A_{k-1}, \ldots, X_0) \in T_{x_0} \mathcal{O}(x_0)$ if and only if

\[
X_{k-1}A_{k-2} - A_{k-1}X_{k-2} + \ldots + X_0A_0 - A_0X_0 = 0.
\]

Let $x_0$ be a polynomial matrix, the values of eigenvalues of all polynomial matrices “near” of $x_0$ are eigenvalues for some polynomial matrix in the versal deformation of $x_0$.

IV. PERTURBATION OF EIGENVALUE OF ARBITRARY MULTICLASS WITH SINGLE EIGENVECTOR

Let $P(\lambda; p) = \lambda^2 + A(p)$ with $A(p) = \begin{pmatrix} -1 & p \\ 0 & 0 \end{pmatrix}$ be a one parameter family of polynomial matrices. The eigenvalues are

\[
\lambda = \pm \sqrt{1 + \sqrt{1 + 4p^2}}
\]

that they are branches of one quadruple-valued analytic function

\[
\lambda(p) = \sqrt{1 + \sqrt{1 + 4p^2}}
\]

the exceptional points are:

- $p = \frac{1}{2}$ and the eigenvalues are $\pm \sqrt{2}$ both double.

- $p = \frac{1}{2}$ and the eigenvalues are $\pm 2$ both double.

- $p = 0$ and the eigenvalues are $0, 1, -1$ both simple and 0 being double.

We observe that for $p = \frac{1}{2}$, the polynomial matrix $P(\lambda; p)$ has a single eigenvalue and a non-zero scaling factor for the double eigenvalue $\lambda = \sqrt{2}$.

We next consider the behavior of the eigenvalues in the neighborhood of one of the exceptional points. Concretely we take $p = \frac{1}{2}$. In this case the eigenvalues are not differentiable functions of the parameter at $p = \frac{1}{2}$, just where the double eigenvalue appears, the eigenvalues tend to infinity as $p$ approaches to $\frac{1}{2}$.

Therefore the analysis of perturbations of multiple eigenvalues with single eigenvalue must be treated in a different manner.

Let $P(\lambda; p)$ be a monic polynomial matrix family and $\lambda_0$ an eigenvalue of arbitrary multiplicity $\ell$ with single eigenvalue up to a non-zero scaling factor at the point $p = p_0$, then, there exists a Jordan chain $v_0, \ldots, v_{\ell-1}$ such that

\[
P(\lambda_0, p_0)v_0 = 0,
\]

\[
P(\lambda_0, p_0)v_0 + P(\lambda_0, p_0)v_1 = 0,
\]

\[
\frac{1}{(\ell - 1)!} P^{\ell - 1}(\lambda_0, p_0)v_0 + \ldots + P(\lambda_0, p_0)v_{\ell-1} = 0,
\]

and, there exists a left Jordan chain $u_0, \ldots, u_{\ell-1}$ such that

\[
u_0^\ell P(\lambda_0, p_0) = 0,
\]

\[
u_0^\ell P(\lambda_0, p_0)+u_1^\ell P(\lambda_0, p_0) = 0,
\]

\[
\frac{1}{(\ell - 1)!} u_0^{\ell - 1} P(\lambda_0, p_0)v_0 + \ldots + u_0^{\ell - 1} P(\lambda_0, p_0)v_{\ell-1} = 0.
\]

Remark 2: a) $u_0^\ell P(\lambda_0, p_0)v_0 = 0,$

b) $u_0^\ell P(\lambda_0, p_0)v_0 = 0 \iff u_1^\ell P(\lambda_0, p_0)v_1 = 0$

c) $u_0^\ell P(\lambda_0, p_0)v_1 = u_1^\ell P(\lambda_0, p_0)v_0$.

In order to analyze the behavior of two eigenvalues $\lambda(p)$ that merge to $\lambda_0$ at $p_0$, we consider a perturbation of the parameter along a smooth curve $p = p(\varepsilon)$, where $\varepsilon \geq 0$ is a small real perturbation parameter and $p(0) = p_0$.

Along the curve $p = p_0 (p_1(\varepsilon), \ldots, p_r(\varepsilon))$ we have a one parameter family $\mathcal{P}(\lambda(x), p)$, which can be represented in the form of Taylor expansion

\[
P(\lambda, p(x)) = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \ldots,
\]

with $P_0 = P(A, p_0)$, $P_1 = \sum_{i=1}^{r} \frac{\partial P(A(x), p(x))}{\partial p_i} \frac{dP_i}{d\varepsilon}$ and the derivatives are evaluated at $p_0$.

Taking into account that $P(\lambda, p(x)) = \sum_{i=0}^{k} \lambda_i A_i(p(x)) (A_0(p(x)) = I_n)$, we have that

\[
P(\lambda, p(x)) = \sum_{i=0}^{k} \lambda_i A_i(p(x))
\]

where $A_k + \varepsilon A_k + \varepsilon^2 A_k + \ldots = I_n$, $A_0 = A_0(p_0)$, $A_1 = \sum_{i=1}^{r} \frac{\partial A_i(p(x))}{\partial p_i} \frac{dP_i}{d\varepsilon}$, and the derivatives are evaluated at $p_0$.

If $\lambda_0$ is a $\ell$-multiplicity eigenvalue of $P(\lambda; p_0)$ having a unique eigenvector $v_0$ up to a non-zero scaling factor the perturbation theory (see [15], for example) tell us that the $\ell$-fold eigenvalue $\lambda_0$ generally splits into $\ell$ of simple eigenvalues $\lambda$ under perturbation of the polynomial matrix $P(\lambda; p_0)$. These eigenvalues $\lambda$ and the corresponding eigenvectors $v$ can be represented in the form of the Puiseux series:

\[
\lambda = \lambda_0 + \varepsilon^{1/\ell} \lambda_1 + \varepsilon^{2/\ell} \lambda_2 + \varepsilon^{3/\ell} \lambda_3 + \ldots
\]

\[
v = v_0 + \varepsilon^{1/\ell} w_1 + \varepsilon^{2/\ell} w_2 + \varepsilon^{3/\ell} w_3 + \ldots
\]

Lemma 1: Let $p_0$ be a point such that $\lambda(p_0) = \lambda_0$ is a $\ell$-multiplicity eigenvalue with single eigenvector $v_0$ and $u_0$ a corresponding left eigenvector. Then, $[v_0]^\ell = \text{Im} P(\lambda_0, p_0)$.

Proof: Let $z \in \text{Im} P(\lambda_0, p_0)$, then there exists a vector $x$ such that $P(\lambda_0, p_0)x = z$. So

\[
u_0^\ell x = u_0^\ell P(\lambda_0, p_0)x = 0 \iff x = 0,
\]

consequently $\text{Im} P(\lambda_0 ; p_0) \subset [v_0]^{\ell}$.

Corollary 1: With the same conditions as the previous lemma, we have

\[
\frac{1}{(\ell - 1)!} u_0^\ell P(\lambda_0, p_0)v_0 + \ldots + u_0^\ell P(\lambda_0, p_0)v_{\ell-1} \neq 0.
\]

Proof: Suppose $\frac{1}{(\ell - 1)!} u_0^\ell P(\lambda_0, p_0)v_0 + \ldots + u_0^\ell P(\lambda_0, p_0)v_{\ell-1} = 0$. Then

\[
\frac{1}{(\ell - 1)!} P(\lambda_0, p_0)v_0 + \frac{1}{(\ell - 1)!} P^{\ell - 1}(\lambda_0, p_0)v_1 + \ldots + P(\lambda_0, p_0)v_{\ell-1} = 0
\]
\[ P(\lambda; p(x)) = (\lambda_0^k I_n + (k-1)\lambda_0^{k-2}A_{k-1} + \ldots + \lambda_1 A_1 + \lambda_0 A_0) + \varepsilon^{1/2}(k\lambda_0^{k-1}A_1 + \ldots + \lambda_2 A_2 + (k-1)\lambda_0^{k-2}A_{k-2} + \ldots + \lambda_1 A_1) + \varepsilon(k\lambda_0^{k-1}A_2 + 1/2(k-1)\lambda_0^{k-2}A_{k-2} + \ldots + \lambda_1 A_1) + (2(k-1)(k-2)\lambda_0^2A_{k-3} + \lambda_2 A_2 A_2 + \lambda_0 A_0 + \ldots + \lambda_0) + \ldots \]

If \( v \) is an eigenvector for the eigenvalue \( \lambda \), we have that

\[ P(\lambda; p(z))v = P(\lambda; p(z))(v + \varepsilon^{1/2}w_1 + \varepsilon w_2 + \ldots) = 0. \]

Then, we find the chain of equations for the unknowns \( \lambda_1, \lambda_2, \ldots \)

\[ P(\lambda_1, \lambda_2; v_0) = 0, \]

\[ \lambda_1 P(\lambda_0; v_0) + P(\lambda_0; p_0)v_0 = 0, \]

\[ P(\lambda_0; p_0)w_2 + \lambda_1 P(\lambda_0; p_0)w_1 + 1/2 \lambda_2^2 P(\lambda_0; p_0)v_0 + \lambda_3 P(\lambda_0; p_0)v_0 + P(\lambda_0; p_0)v_0 = 0, \]

\[ \lambda_2 P(\lambda_0; p_0)w_2 + \lambda_1 P(\lambda_0; p_0)w_1 + P(\lambda_0; p_0)v_0 + \lambda_3 P(\lambda_0; p_0)v_0 = 0, \]

\[ \lambda_3 P(\lambda_0; p_0)v_0 = 0, \]

where \( P(\lambda_1; p_0) = \lambda_0^{k-1}A_{k-1} + \lambda_0 A_{k-2} + \ldots + \lambda_0 A_1 + \lambda_0 A_0. \)

Equation (18) is satisfied because \( v_0 \) is an eigenvector corresponding to the eigenvalue \( \lambda_0 \). Comparing equation (32) with (3) for \( i = 1 \), we observe that \( w_1 = \lambda_1 v_1 + \beta v_0 \) for all \( \beta \) a solution, we take

\[ w_1 = \lambda_1 v_1. \]

To find the value of \( \lambda_1 \), we prewmiti equation (20) by \( u_0^0 \), using the given value for \( w_1 \) and taking into account \( u_0^0 P(\lambda_0; p_0)v_0 = 0 \) and \( u_0^0 P(\lambda_0; p_0)v_0 = 0 \) we obtain

\[ \lambda_1^2 P(\lambda_0; p_0)v_1 + 1/2 \lambda_2^2 P(\lambda_0; p_0)v_0 + u_0^0 P(\lambda_0; p_0)v_0 = 0. \]

Taking into account corollary 1 we can find

\[ \lambda_1 = \pm \frac{-u_0^0 P(\lambda_0; p_0)v_0}{u_0^0 P(\lambda_0; p_0)v_1 + 1/2 u_0^0 P(\lambda_0; p_0)v_0}. \]

If \( u_0^0 P(\lambda_0; p_0)v_0 \neq 0 \) we have two values of \( \lambda_1 \) that determine leading terms in expansions for two different eigenvalues \( \lambda \) that bifurcate from the double eigenvalue \( \lambda_0 \).

Suppose then, that \( u_0^0 P(\lambda_0; p_0)v_0 \neq 0 \). Premultiplying (21) by \( u_0^0 \),

\[ \lambda_1 u_0^0 P(\lambda_0; p_0)v_0 + 1/2 \lambda_2 u_0^0 P(\lambda_0; p_0)v_0 + \lambda_3 u_0^0 P(\lambda_0; p_0)v_0 + \lambda_1 P(\lambda_0; p_0)v_0 = 0. \]

Premultiplying (20) by \( u_1^0 \), and according to 2, we have:

\[ u_0^0 P(\lambda_0; p_0)v_0 = \lambda_1 u_0^0 P(\lambda_0; p_0)v_0 + 1/2 \lambda_2 u_0^0 P(\lambda_0; p_0)v_0 + \lambda_0 u_0^0 P(\lambda_0; p_0)v_0 + u_0^0 P(\lambda_0; p_0)v_0 = 0. \]

Now, we can compute \( w_2 \). We have

\[ P(\lambda_0; p_0)v_0 = -\lambda_1 P(\lambda_0; p_0)v_0 - 1/2 \lambda_2^2 P(\lambda_0; p_0)v_0 - \lambda_3 P(\lambda_0; p_0)v_0 + P(\lambda_0; p_0)v_0. \]

Lemma 2: Following condition \( u_0^0 P(\lambda_0; p_0)v_0 \neq 0 \) we have that

\[ P(\lambda_0; p_0) + u_0^0 u_1^0 P(\lambda_0; p_0)v_0 = 0 \]

is an invertible matrix.

Proof: Let \( x = \alpha v_0 + w \) with \( w \in [v_0]^\perp \), be a vector in the null space, then \( P(\lambda_0; p_0) + u_0^0 u_1^0 P(\lambda_0; p_0)v_0 = 0 \).

Premultiplying by \( u_0^0 \) we have

\[ u_0^0 (P(\lambda_0; p_0) + u_0^0 u_1^0 P(\lambda_0; p_0)v_0)u_0^0 x = 0, \]

\[ = 0 = u_0^0 u_1^0 P(\lambda_0; p_0)v_0(\alpha v_0 + w) = |\alpha||u_0||u_0^0||v_0^0||u_1^0 P(\lambda_0; p_0)v_0. \]

Then \( \alpha = 0 \).

Consequently, \( x = w \in [v_0]^\perp \) and \( x \in \ker u_0^0 u_1^0 P(\lambda_0; p_0)v_0 \), so \( x \in \ker P(\lambda_0; p_0) \) and \( x = \beta v_0 \), but \( x \in [v_0]^\perp \), then \( \beta = 0 \).
B. Perturbation of a \(\ell\)-multiplicity eigenvalue with single eigenvector

Now, we analyze the general case. Analogously, substituting (16) into (15) we obtain

\[
P(\lambda; p(z)) = \left(\lambda_0 + \varepsilon^{1/\ell} \lambda_1 + \cdots + \varepsilon^{k/\ell} \lambda_k I_n + (\lambda_0 + \varepsilon^{1/\ell} \lambda_1 + \cdots)^{-1}(A_{k-1} + \cdots + \varepsilon^{k-1} A_{k-1} + \cdots) + \cdots + (\lambda_0 + \varepsilon^{1/\ell} \lambda_1 + \varepsilon^{2/\ell} \lambda_2 + \cdots)(A_{1} + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots) + A_{0} + \varepsilon A_{0} + \varepsilon^2 A_{0} + \cdots = \right.
\]

\[
\left(\lambda_0^\ell + \lambda_0^{\ell-1} \lambda_1 + \cdots + \lambda_0 A_{0} + A_{0} + \varepsilon^{1/\ell} (k \lambda_0^{\ell-1} \lambda_1 I_n + (k-1) \lambda_0^{\ell-2} \lambda_1 A_{k-1} + \cdots + \lambda_0 A_{1}) + \right.
\]

\[
\left.\varepsilon^{2/\ell}((k \lambda_0^{\ell-1} \lambda_2 + \lambda_0 A_{1} A_{0}) + \cdots + \frac{1}{2} (k-1) (k-2) \lambda_0^{\ell-2} \lambda_2 + \cdots + \lambda_0 A_{1} I_n) + \cdots \right)
\]

Remark 3: If \(\varepsilon\) is small and \(\lambda_0 \neq \lambda_1 \neq \cdots \neq \lambda_k\), then, the eigenvalue \(\lambda_0\) of pencil (15) is simple as eigenvalue of \(A\), and the corresponding eigenvector \(v_0\) is a left eigenvector of the matrix \(A^T\).

Proof: If \(A^T v_0 = \lambda_0 v_0\), then, the eigenvalue \(\lambda_0\) of pencil (15) is simple as eigenvalue of \(A\), and the corresponding eigenvector \(v_0\) is a left eigenvector of the matrix \(A^T\).

Remark 4: If \(\varepsilon\) is small and \(\lambda_0 \neq \lambda_1 \neq \cdots \neq \lambda_k\), then, the eigenvalue \(\lambda_0\) of pencil (15) is simple as eigenvalue of \(A\), and the corresponding eigenvector \(v_0\) is a left eigenvector of the matrix \(A^T\).

Definition 1: An eigenvalue \(\lambda_0\) of a system \((A, B)\) is called simple if it is simple as eigenvalue of \(A\).

Observe that an eigenvalue of \(A\) is not necessarily an eigenvalue of \((A, B)^T\).

Proposition 5 ([113]): Let \(\lambda_0\) be a simple eigenvalue of \((A, B)\). Then \(\lambda_0\) is an eigenvalue of \((A, B)^T\) with the corresponding eigenvector of \((A, B)\).

Proof: If \(A^T v_0 = \lambda_0 v_0\) and \(B^T v_0 = 0\) then \(K^T B^T v_0 = 0\) and \((A^T + K^T B^T) v_0 = \lambda_0 v_0\).

Reciprocally, if \((A^T + K^T B^T) v_0 = \lambda_0 v_0\) then \(A^T v_0 + K^T B^T v_0 = (A^T + K^T B^T) v_0 = \lambda_0 v_0\).

Corollary 2: Let \(K\) such that \(\mu_0\) is an eigenvalue of \((A^T + K^T B^T)\) and \(v_0\) a corresponding eigenvector. If \(\mu_0\) is not an eigenvalue of \((A, B)^T\), then \(B^T v_0 \neq 0\).

Let \((A, B)\) be a linear system and we assume that the matrices \(A, B\) smoothly depend on the vector of a real parameters \(p = (p_1, \ldots, p_n)\). The function \((A(p), B(p))\) is called a multi-parameter family of linear systems. Eigenvalues of linear system function are continuous functions \(\lambda(p)\) of the vector of parameters. In this section we are going to study the behavior of a simple eigenvalue of the family of linear systems \((A(p), B(p))\).

Let us consider a point \(p_0\) in the parameter space and assume that \(\lambda(p_0) = \lambda_0\) is a simple eigenvalue of \((A(p_0), B(p_0)) = (A_0, B_0)\), and \(v(p_0) = v_0\) an eigenvector, i.e.

\[
A_0 v_0 = \lambda_0 v_0, \quad B_0 v_0 = 0.
\]

Equivalently

\[
(A_0 + K^T B^T) v_0 = \lambda_0 v_0, \quad B_0^T v_0 = 0, \quad \forall K.
\]

Now, we are going to review the behavior of a simple eigenvalue \(\lambda(p)\) of the family of standard linear systems.

The eigenvector \(v(p)\) corresponding to the simple eigenvalue \(\lambda(p)\) determines a one-dimensional null-subspace of the matrix operator \(A^T\) smoothly dependent on \(p\). Hence, the eigenvector \(v(p)\) can be chosen as a smooth function of the parameters. We will try to obtain an approximation by means their derivatives.

We write the eigenvalue problem as

\[
\begin{align*}
(A^T(p) + K^T B^T(p)) v(p) &= \lambda(p) v(p) \\
B^T(p) v(p) &= 0.
\end{align*}
\]

Now, corollary 1 ensures the result.
equivalently
\[ A'(p)v(p) = \lambda(p)v(p) \]
\[ B'(p)v(p) = 0 \quad \] (31)

Taking the derivatives with respect to \( p_i \)
\[ \frac{\partial A'(p)}{\partial p_i} v(p) + A'(p) \frac{\partial v(p)}{\partial p_i} = \frac{\partial \lambda}{\partial p_i} v(p) + \lambda(p) \frac{\partial v(p)}{\partial p_i} \]
\[ \frac{\partial B'(p)}{\partial p_i} v(p) + B'(p) \frac{\partial v(p)}{\partial p_i} = 0 \quad \] (32)

At the point \( p_0 \) we have.
\[ \left( \frac{\partial A'(p)}{\partial p_i} - \frac{\partial A'(p)}{\partial p_i} I_n \right) v_0 = (\lambda I_n - A'(p_0)) \frac{\partial v(p)}{\partial p_i} \]
\[ \frac{\partial B'(p)}{\partial p_i} v_0 + B'(p_0) \frac{\partial v(p)}{\partial p_i} = 0 \quad \] (33)

This is a linear algebraic system of equations for the unknowns \( \frac{\partial \lambda}{\partial p_i} \) and \( \frac{\partial v(p)}{\partial p_i} \) where the matrix \( \lambda I_n - A'(p_0) \) is singular with rank equal \( n - 1 \) because of \( \lambda_0 \) is a simple eigenvalue.

**Lemma 3 ([13]):** The matrix \((\lambda I_n - A'(p_0) - u_0u_0^T)\) is invertible.

**Proposition 7:** With the same conditions, the system (32) has a solution if and only if
\[ u_0' \frac{\partial \lambda}{\partial p_i} I_n - A'(p_0) v_0 = 0 \]
\[ \frac{\partial B'(p)}{\partial p_i} v_0 + B'(p_0) \frac{\partial v(p)}{\partial p_i} = 0 \quad \] (34)

where \( u_0 \) is a left eigenvector for the simple eigenvalue \( \lambda_0 \) of the matrix \( A'. \)

**Proof:** From first equation of (33) we obtain a solution for \( u_0' \frac{\partial \lambda}{\partial p_i} : \)
\[ \frac{\partial \lambda}{\partial p_i} (u_0'v_0) = u_0' \frac{\partial A'(p_0)}{\partial p_i} v_0. \]
\[ \frac{\partial \lambda}{\partial p_i} = u_0' \frac{\partial A'(p_0)}{\partial p_i} v_0. \]

We can choose \( u_0 \) in such away that \( u_0'v_0 = 1 \)

Replacing this solution in first equation of (32) we obtain
\[ \frac{\partial v(p)}{\partial p_i} = (\lambda I_n - A'(p_0) - u_0u_0^T)^{-1} \left( \frac{\partial A'(p_0)}{\partial p_i} - \frac{\partial \lambda}{\partial p_i} I_n \right) v_0. \]

Now we need to see if this expression verifies the second equation of (32).

Taking the partial derivative \( \frac{\partial^2}{\partial p_i \partial p_j} \) of both sides of eigenvalue problem (30) we have:
\[ \left\{ \begin{array}{l}
\frac{\partial^2 A'(p)}{\partial p_i \partial p_j} v(p) + \frac{\partial A'(p)}{\partial p_i} \frac{\partial v(p)}{\partial p_j} + \frac{\partial A'(p)}{\partial p_j} \frac{\partial v(p)}{\partial p_i} + A'(p) \frac{\partial^2 v(p)}{\partial p_i \partial p_j} = \frac{\partial^2 \lambda(p)}{\partial p_i \partial p_j} v(p) + \frac{\partial \lambda(p)}{\partial p_i} \frac{\partial v(p)}{\partial p_j} + \frac{\partial \lambda(p)}{\partial p_j} \frac{\partial v(p)}{\partial p_i} + \frac{\partial^2 v(p)}{\partial p_i \partial p_j} \\
\frac{\partial^2 B'(p)}{\partial p_i \partial p_j} v(p) + \frac{\partial B'(p)}{\partial p_i} \frac{\partial v(p)}{\partial p_j} + \frac{\partial B'(p)}{\partial p_j} \frac{\partial v(p)}{\partial p_i} + B'(p) \frac{\partial^2 v(p)}{\partial p_i \partial p_j} = 0
\end{array} \right. \]

At \( p_0 \) and premultiplying the equation by \( u_0' \) we can deduce an expression for derivatives \( \frac{\partial^2 \lambda(p)}{\partial p_i \partial p_j} \)
\[ \frac{\partial^2 \lambda(p)}{\partial p_i \partial p_j} u_0'v_0 = u_0' \frac{\partial^2 A'(p)}{\partial p_i \partial p_j} v_0 + u_0' \frac{\partial A'(p)}{\partial p_i} \frac{\partial v(p)}{\partial p_j} + u_0' \frac{\partial A'(p)}{\partial p_j} \frac{\partial v(p)}{\partial p_i} - u_0' \frac{\partial \lambda(p)}{\partial p_i} \frac{\partial v(p)}{\partial p_j} - u_0' \frac{\partial \lambda(p)}{\partial p_j} \frac{\partial v(p)}{\partial p_i} \]

Knowing \( \frac{\partial^2 \lambda(p)}{\partial p_i \partial p_j} \) we can deduce the values of \( \frac{\partial^2 v(p)}{\partial p_i \partial p_j} \)

calling \( S = (A'(p) - \lambda(p)I - u_0u_0^T)^{-1} \)
\[ \frac{\partial^2 v(p)}{\partial p_i \partial p_j} = \frac{\partial^2 \lambda(p)}{\partial p_i \partial p_j} \frac{\partial v(p)}{\partial p_j} + \frac{\partial \lambda(p)}{\partial p_i} \frac{\partial v(p)}{\partial p_j} + \frac{\partial \lambda(p)}{\partial p_j} \frac{\partial v(p)}{\partial p_i} - \frac{\partial^2 v(p)}{\partial p_i \partial p_j} \]

VI. PERTURBATION ANALYSIS OF SIMPLE EIGENVALUES OF SINGULAR SYSTEMS

Finally, we consider systems in the form \( E \dot{x} = Ax + Bu \) with \( E, A \in M_n(\mathbb{C}) \) and \( B \in M_{n \times m}(\mathbb{C}) \), we will write the systems as a triple of matrices \((E, A, B)\).

Let \( M(\lambda) = (AE + AB) \) be a matrix pencil associated to the triple \((E, A, B)\), \( \lambda_0 \) is an eigenvalue of \((E, A, B)\), if rank \( M(\lambda_0) < \) rank \( M(\lambda) \). In the case where the matrix pencil \( AE + AB \) is regular this is equivalent to \( \det(\lambda E + A) = 0 \).

\( v_0 = \mathbb{C}^n \) is an eigenvector corresponding to the eigenvalue \( \lambda_0 \), if \( (\lambda_0 E + A)^{-1} v_0 = 0 \) and \( B'v_0 = 0 \).

**Proposition 8:** Let \( \lambda_0 \) be an eigenvalue and \( v_0 \) a corresponding eigenvector of \((E, A, B)\). Then \( \lambda_0 \) is an eigenvalue and \( v_0 \) the corresponding eigenvector of \((E + BK_1, A + BK_2, B)\) for all \( K \).

Suppose that matrices \( E, A, B \) defining the singular system, smoothly depend on the vector of a real parameters \( p = (p_1, \ldots, p_k) \).

The function \( E(p), A(p), B(p) \) is called a multi-parameter family of singular systems.

We write the eigenvalue problem as
\[ \left\{ \begin{array}{l}
(\lambda E'(p) + A'(p))v(p) = 0 \\
B'(p)v(p) = 0
\end{array} \right. \]

equivalently
\[ \left\{ \begin{array}{l}
(\lambda E'(p) + A'(p))v(p) = 0 \\
B'(p)v(p) = 0
\end{array} \right. \]

Taking derivatives
\[ \left\{ \begin{array}{l}
\frac{\partial \lambda}{\partial p_i} E'(p) + \lambda \frac{\partial E'(p)}{\partial p_i} + \frac{\partial A'(p)}{\partial p_i} v(p) \\
\frac{\partial B'(p)}{\partial p_i} v(p) + B'(p) \frac{\partial v(p)}{\partial p_i} = 0
\end{array} \right. \]

At the point \( (\lambda_0, p_0) \)
\[ \left\{ \begin{array}{l}
\frac{\partial \lambda}{\partial p_i} E'(p) + \lambda \frac{\partial E'(p)}{\partial p_i} + \frac{\partial A'(p)}{\partial p_i} v(p) \\
\frac{\partial B'(p)}{\partial p_i} v(p) + B'(p) \frac{\partial v(p)}{\partial p_i} = 0
\end{array} \right. \]
Prelimiting by $u_0$ the first equality we have
\[
\begin{align*}
&\left. u_0 \left( \frac{\partial \lambda}{\partial p_1} E'(p) + \lambda \frac{\partial E'(p)}{\partial p_1} + \frac{\partial A'(p)}{\partial p_1} \right) \right| \left. _{\lambda_0(p_0)} = 0 \right) \\
&\left. \frac{\partial B'(p)}{\partial p_1} \right| \left. _{\lambda_0(p_0)} = 0 \\
&\left. \frac{\partial v}{\partial p_1} \right| \left. _{\lambda_0(p_0)} = 0 \\
\end{align*}
\]
Using the normalization condition $\frac{\partial v}{\partial p_1}(p) = 1$ (it is possible because the function $u_0v(p)$ in $p = p_0$ is non zero) we have that $u_0 \frac{\partial v}{\partial p_1}(p_0) = 0$.

**Lemma 4**: There exists a left eigenvector such that $u_0' E(p_0)v_0 \neq 0$.

**Proof**: Taking into account that $\lambda_0$ is a simple eigenvalue $E'(p_0) + (\lambda_0 E')^\perp v_0 \neq 0$ for all vector $v_1$. Taking $v_1 = 0$ we have that $E'(p_0)v_0 \neq 0$.

If $u_0 E'(p_0)v_0 = 0$ we have that $E(p_0, A_0v_0) \in [v_0]^\perp$, so $u_0$ is an eigenvector of the linear map $(\lambda_0 E + A)v_0 \in [v_0]^\perp$. For the zero eigenvalue of, but zero is a simple eigenvalue of $\lambda_0 E + A$.

**Lemma 5**: The matrix $T_0 = \lambda_0 E'(p_0) + A'(p_0) + u_0 u_0'^\perp$ is invertible.

**Proof**: $u_0 u_0'$ is a symmetric map of rank 1, $u_0$ is an eigenvector of eigenvalue $\|u_0\|^2$ and $[u_0]^\perp$ is the null-space. Let $w \in \text{Ker} T_0$, we can write $w = \alpha u_0 + w_1$ with $w_1 \in [u_0]^\perp$. Then $0 = T_0w$ and
\[
0 = u_0' T_0 w = u_0' (\lambda_0 E'(p_0) + A'(p_0) + u_0 u_0')(\alpha u_0 + w_1) = u_0' (\alpha u_0 u_0') (\alpha u_0 + w_1) = \alpha (u_0 u_0')^2.
\]
Then $\alpha = 0$ and $w = w_1 \in \text{Ker} u_0 u_0'$, consequently $(\lambda_0 E'(p_0) + A'(p_0)w_1) = 0$, and taking into account that $\lambda_0$ is a simple eigenvalue we have $w_1 = \beta v_0 \in [v_0]^\perp$. Finally, condition $u_0' E(p_0) v_0 \neq 0$ implies $\beta = 0$ and $T_0$ is invertible.

**VII. Conclusion**

In this paper the perturbation of a multiple eigenvalue with a simple eigenvector of a monic polynomial matrix smoothly depending on parameters is analyzed, as well as the perturbation of a simple eigenvalue of a standard and a singular linear system smoothly depending on parameters.

**References**


